Capacity of the MIMO Channel

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Overview

- Channel Capacity
- AWGN Channel
- Fading Channel
- Spatial Multiplexing and Diversity
- MIMO Channel

Channel Capacity

Definitions:

Mutual Information: A measure of the mutual dependence between two variables

$$I(X;Y) = H(X) - H(Y|X)$$

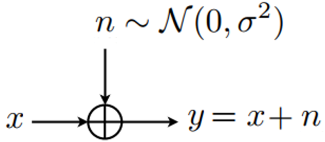
Channel Capacity: The maximum rate of communication on that channel for which the error probability is arbitrarily small

$$C = \max_{p(x)} I(X;Y)$$

where X,Y are the input and output random variables respectively, and $p(x)$ is the input distribution

AWGN Channel

- Channel model:



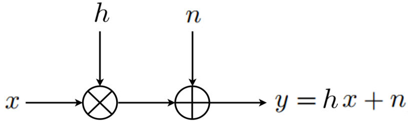
where the input $x[n]$ has limited power P

- AWGN Channel capacity:

$$C = \frac{1}{2} \log(1 + \frac{P}{\sigma^2})$$

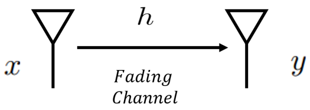
Fading Channels

- We now incorporate fading to our channel model:
  - What is fading ?
  - Flat vs Frequency Selective Fading
  - Slow vs Fast Fading



- here n is a complex AWGN, and y is a complex output
- h is the channel impulse response that characterizes fading

Capacity of Fading Channels



- We first derive the capacity of the single input single output (SISO) fading channel
- Assumptions:
  - A flat fading channel
  - The channel is perfectly known at the receiver
  - The input X is limited in power to P_t
  - The noise N is independent of the input X: $H(N|X) = H(N)$

Capacity Derivation

$$
\begin{aligned}
C &= \max_{p(x), P \leq P_t} I(X;Y) \\
&= \max_{p(x), P \leq P_t} H(Y) - H(Y|X) \\
&= \max_{p(x), P \leq P_t} H(Y) - H(hX + N|X) \\
&= \max_{p(x), P \leq P_t} H(Y) - H(N)
\end{aligned}
$$

- For a complex Gaussian noise, the entropy is known to equal $\log(\pi e \sigma_n^2)$
- Above, the mutual information is maximized by maximizing $H(Y)$
- The entropy is maximized if the variable Y is also Gaussian $\Rightarrow H(Y) = \log(\pi e \sigma_y^2)$

Capacity Derivation (continued)

- The variance σ_y^2 (received average power)can be found as :

$$\sigma_y^2 = E[Y^2] = E[(hX+N)(hX+N)^*] = \sigma_x^2|h|^2 + \sigma_n^2$$

- The capacity can now be derived as:

$$\begin{aligned} C &= H(Y) - H(N) \\ &= \log(\pi e \sigma_y^2) - \log(\pi e \sigma_n^2) \\ &= \log(\pi e[\sigma_x^2|h|^2 + \sigma_n^2]) - \log(\pi e \sigma_n^2) \\ &= \log(1 + \frac{\sigma_x^2}{\sigma_n^2}|h|^2) \\ &= \log(1 + \frac{P_t}{\sigma_n^2}|h|^2) \end{aligned}$$

Ergodic Capacity

$$C_{fading} = \log(1 + \frac{P_t}{\sigma_n^2}|h|^2)$$

- The channel impulse response h is a random variable $\rightarrow C_{fading}$ is also random
- Define the statistical average of the capacity, with expectation over $|h|^2$ as the Ergodic capacity:

$$C_{ef} = E[\log(1 + \frac{P_t}{\sigma_n^2}|h|^2)]$$

Jensen's Inequality

- Jensen's inequality states:

$$E[f(X)] \leq f(E[X])$$

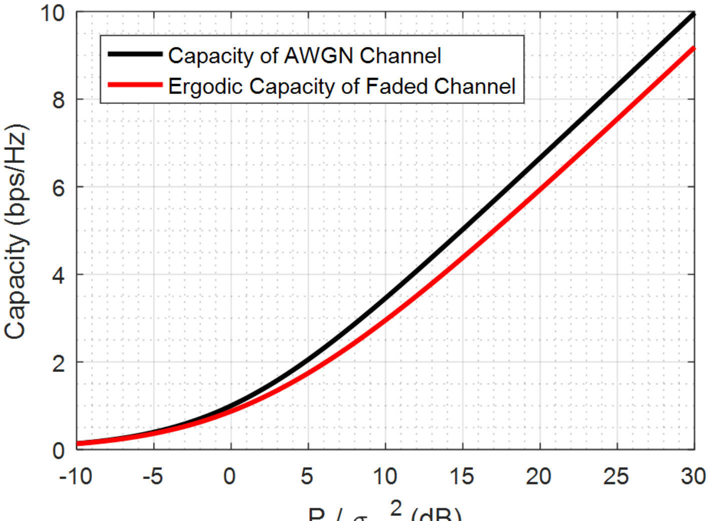
- Applying it to C_{fading}, with expectation over the impulse response h we get :

$$E[\log(1 + \frac{P_t}{\sigma_n^2}|h|^2)] \leq \log(1 + \frac{P_t}{\sigma_n^2}E[|h|^2])$$

- This implies that the Ergodic fading capacity cannot exceed that of an AWGN channel with constant gain
- Fading is bad!

Result Visualization

- Using a Rayleigh distribution with $\sigma_h^2 = 1$ to characterize the fading, and $\sigma_n^2 = 1$ we simulate and visualize the above result



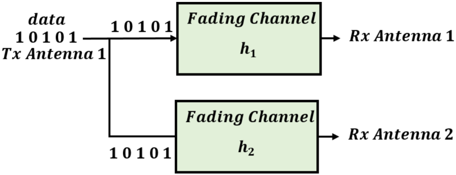
Spatial Multiplexing and Diversity

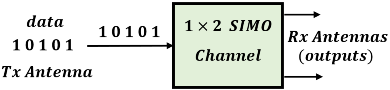
How does MIMO improve performance?

- Diversity mitigates fading
- Spatial Multiplexing improves data rate

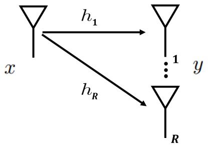
Diversity

- Same information is sent across independent fading channels
- Each channel will suffer independent amount of fading
- Chance of proper delivery increases
- At least one link will suffer less fading
- This improves reliability and reduces co channel interference





SIMO



Lets consider the capacity of the single input multiple output channel (SIMO) without fading, when:

$$\mathbf{y} = \mathbf{h}x + \mathbf{w}$$

through Maximum Ratio Combining (MRC) :

$$\mathbf{y} = \hat{y}\frac{\mathbf{h}}{||\mathbf{h}||}$$

$$\hat{y} = ||\mathbf{h}||x + \hat{w}$$

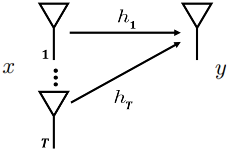
SIMO continued

this is a scalar AWGN channel with a a capacity:

$$C = \log(1 + \frac{P_t}{\sigma_n^2}||h||^2)$$

- Since $||\mathbf{h}||^2$ is a constant, there is an increase in the SNR of the equivalent AWGN SISO channel
- The term $||\mathbf{h}||^2$ is an increase to the SNR and considered a power gain

MISO



Lets consider the capacity of the multiple input single output channel (MISO) without fading, when:

$$y = \mathbf{h} * \mathbf{x} + w$$

through Maximum Ratio Combining (MRC) :

$$\mathbf{x} = \hat{x}\frac{\mathbf{h}}{||\mathbf{h}||}$$

$$y = ||\mathbf{h}||\hat{x} + w$$

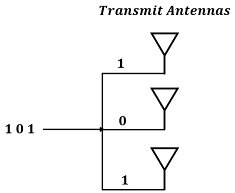
MISO continued

this is a scalar AWGN channel with a a capacity:

$$C = \log(1 + \frac{P}{\sigma_n^2}||h||^2)$$

- The capacity of MISO and SIMO are similar, with a power constraint on the input
- In both cases we observe power gain through beamforming

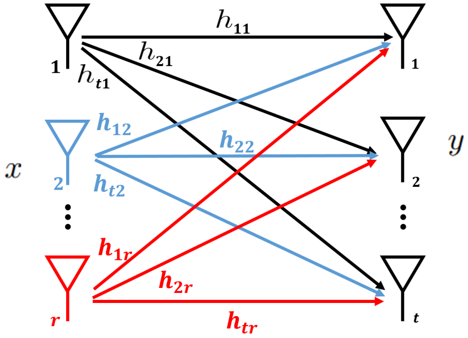
Spatial Multiplexing



- Each spatial channel carries independent information $\rightarrow$ increasing the data rate of the system
- If scattering is present, several independent sub channels are created resulting in multiplexing gain

MIMO Channel

- Multiple Input Multiple Output fading channel with t transmitting and r receiving antennas:



MIMO Channel

- The received vector **y** can be expressed in terms of the channel transmission matrix **H**, the input vector **x** and the noise vector **n** as follows:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

- where:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} \quad H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & h_{r2} & \cdots & h_{rt} \end{bmatrix} \quad n = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_t \end{bmatrix}$$

MIMO Channel

Definition of variables:

- N_t - number of transmit antennas
- N_r - number of receive antennas
- **y** - received response from the channel - $(N_r \times 1)$
- **H** - complex channel matrix - $(N_r \times N_t)$
- **x** - transmitted signal vector - $(N_t \times 1)$
- **n** - complex AWGN vector - $(N_r \times 1)$
- $\mathbf{K_x} = E[xx^H]$ - covariance matrix of input vector
- $\mathbf{K_y} = E[yy^H]$ - covariance matrix of output vector
- $\mathbf{K_n} = E[nn^H]$ - covariance matrix of noise vector

MIMO Channel

Assumptions:

- Receiver posses perfect knowledge about the channel
- The input random variable **x** is independent from the noise **n**
- **y**,**n** are zero mean and are Gaussian distributed with variance $\mathbf{K_y}, \mathbf{K_n}$ respectively
- The average power that can be expensed at the transmitter is P_t
- The channel is discrete and memoryless

Capacity Derivation

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\begin{aligned}
C &= \max_{p(x), P \le P_t} I(X;Y) \\
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&= \max_{p(x), P \le P_t} H(Y) - H(HX + N|X) \\
&= \max_{p(x), P \le P_t} H(Y) - H(N)
\end{aligned}
$$

- from the definition of entropy, $H(N)$ is found to be $= \log(det[\pi e K_n])$
- also, $H(Y)$ is found to be $= \log(det[\pi e K_y])$
- $H(Y)$ is maximized if y is circularly symmetric complex Gaussian $\rightarrow$ x must be circularly symmetric complex Gaussian (Teletar proves this)

Capacity Derivation continued

with K_y found as follows:

$$\begin{aligned} K_y = E[YY^H] &= E[(HX+N)(HX+N)^H] \\ &= HK_xH^H + K_n \end{aligned}$$

the capacity is given by

$$\begin{aligned} C &= H(Y) - H(N) \\ &= \log(det[\pi e(HK_xH^H + K_n)]) - \log(det[\pi e K_n]) \\ &= \log(det[(HK_xH^H + K_n)]) - \log(det[K_n]) \\ &= \log(det[(HK_xH^H + K_n)(K_n)^{-1}]) \\ &= \log(det[(HK_xH^H)(K_n)^{-1} + I_{N_r}]) \\ &= \log(det[I_{N_r} + (K_n)^{-1}(HK_xH^H)]) \end{aligned}$$

Capacity Derivation continued

- For the case where the noise is uncorrelated between the antenna branches (spatially white)

$$K_n = I_{N_r}$$

- In this case the MIMO flat fading channel capacity is

$$C = \log(det[I_{N_r} + (HK_xH^H)])$$

Capacity Derivation continued

Theorem: When x is circularly symmetric complex Gaussian with zero mean and covariance (P/t)I_t, the ergodic capacity is given by:

$$C = E[\log(det[I_{N_r} + \frac{P}{N_t}(HH^H)])]$$

Capacity Derivation continued

Theorem: When x is circularly symmetric complex Gaussian with zero mean and covariance (P/t)I_t, the ergodic capacity is given by:

$$C = E[\log(det[I_{N_r} + \frac{P}{N_t}(HH^H)])]$$

MIMO Implementations

We can choose how to encode and decode our information when using MIMO

- To improve reliability - implement diversity techniques (e.g. Alamouti Scheme)
- To improve data rate - implement spatial-multiplexing gain (e.g V-Blast)

MIMO Implementations

We can choose how to encode and decode our information when using MIMO

- To improve reliability - implement diversity techniques (e.g. Alamouti Scheme)
  - The Alamouti scheme is usually used with 2x2 antenna systems
  - It send two symbols and their conjugate in two time slots, which brings diversity without compromising on data rate
  - It achieves maximum diversity gain

MIMO Implementations

- To improve data rate - implement spatial-multiplexing gain (e.g V-Blast)
  - A Single data stream divided into substreams
  - The substreams are independently encoded into symbols, and multiplexed through some coordinate system
  - A vertical blast DSP algorithm is used to estimate and decode the received signal
  - Achieves the channel capacities upper bound

Questions